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Phase space representation of a spin-$\frac{1}{2}$ particle interacting with an external electromagnetic field: I. General examination

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Abstract. A method for obtaining phase space representations of the current, the energy-momentum and the spin tensors by means of Fourier expansions of wavefunctions is developed. In the case of spin-$\frac{1}{2}$ fields, it leads to a new type of phase space energy-momentum tensor in physics: a symmetrized product of the canonical momentum and the charge current. The Gordon's decomposition of the current into convective and spin components separates four terms in the phase space energy-momentum tensor: a corpuscular one, and three others describing the interactions of the wavevectors and the external electromagnetic field; the wavevectors and the helicity four-vectors; the helicity vectors and the electromagnetic field, respectively. The phase space representation of the spin tensor shows an unexpected dependence of this tensor on the metric and the charge current. A possible interpretation of the negative values of the phase space current is suggested. Its application to the phase space energy-momentum tensor together with the reinterpretation principle in special relativity leads to a partition of the spin-$\frac{1}{2}$ particle into bradyons, luxons, tachyons and their anti-particles.

1. Introduction

The exploration of quantum distribution functions in phase space began with the discovery of the famous Wigner distribution function [1]. Many other quantum distribution functions have since been found for the purposes of different applications of quantum mechanics [2–12]. The development of the basic ideas, the classification and the main tendencies of the investigations in the field of the quantum distribution functions have been traced out in many reviews [13–24].

The introduction of the Wigner quantum distribution function for particles with spin-$\frac{1}{2}$ has been discussed by a number of authors [25–34]. These investigations have been carried out mainly for systems constituted from many particles. De Groot *et al* [30] showed that it was possible to represent the energy-momentum tensor of a system of non-interacting particles by the second moment of the Wigner distribution function $F_W(x, k)$:

$$T_{\mu\nu}(x) = \text{constant tr} \int k_\mu k_\nu F_W(x, k)\, \mathrm{d}^4 k \qquad \mu, \nu = 0, 1, 2, 3 \tag{1.1}$$

where $(x) = (x^\mu)$ denotes the position in Minkowski space and $k = (k_\mu)$ is the wavevector related to the four-momentum p_μ by $k_\mu = p_\mu/\hbar c$, $\mu = 0, 1, 2, 3$; $\hbar$ is the reduced Planck constant and c is the speed of light.

All the integrations in this work are from $-\infty$ to $+\infty$.

For interpretation purposes in the sequel we accept Recami and Rodrigues [35] concept that special relativity is based on the whole proper group of both ortho- and anti-orthocronous Lorentz transformations, i.e. that in special relativity, particles as well as anti-particles are included.

One can overcome the problem with the interpretation of the negative values of the quantum distribution functions in the following way: any non-positive function $F(x, k)$ can be decomposed into two strictly positive functions by the equality [36]:

$$F(x, k) = F^+(x, k) - F^-(x, k). \tag{1.2}$$

Here the functions

$$F^{\pm}(x, k) = \{|F(x, k)| \pm F(x, k)\}/2 \geqslant 0 \tag{1.3}$$

are always positive. They are continuous and integrable if $F(x, k)$ is smooth and integrable.

Following the propositions in [36] and [37] one can interpret $F^{\pm}(x, k)$ as phase space densities of pairs of random physical quantities (η^+, η^-). This suggestion for $F^{\pm}(x, k)$ could be related to Krügers idea [38] to consider joint distribution functions in the sense of the classical probability theory of a stochastic variable.

Vigier and Terletsky [39] have already proposed a similar decomposition of the probability density of a system into a difference of two positive probability densities corresponding to particles with opposite charges. The representation (1.2) appears as a generalization of their concept to the case of the phase space distribution functions.

Using the quantum distribution functions one wants to 'investigate the description of quantum observables by functions on phase space' [40]. In particular, the aim of the present work is to consider the possibility of describing a spin-$\frac{1}{2}$ particle in an external electromagnetic field by means of suitable representations of its characteristic observables such as the charge current, the energy-momentum and the spin tensors, into the phase space.

The difference of the presented approach from the ones developed in [25–34] is that here the Terletsky-type distribution function [3] is used. Dealing with such a type of a distribution function one can introduce it in the invariants of the quantum field theory by means of the Fourier transforms of the wavefunctions. This possibility seems more natural, both from the mathematical and physical point of view, than the possibility applied in [28, 30] where the Wigner-type distribution functions have been brought in the invariants employing an integration over a four-dimensional δ-function.

The main results of the present work are: (i) The energy-momentum tensor of a spin-$\frac{1}{2}$ particle in phase space is expressed by means of a symmetrized product of the canonical momentum and the phase space density of its charge current. (ii) Applying Gordon's decomposition of the charge current into convective and spin parts the energy-momentum tensor of the spin-$\frac{1}{2}$ particle separates into four terms: a pure corpuscular contribution; a direct interaction between the wave vectors and the external electromagnetic fields; an interaction between the wavevectors and the helicity four-vectors and an interaction between the external electromagnetic field and the helicity four-vectors. (iii) The phase space representation of the spin tensor shows an unexpected dependence on the charge current in phase space and the metric tensor. Here the Gordon's decomposition of the current separates again the phase space spin tensor in four terms: a term connected with the interaction between the wavevectors and the metric; a term demonstrating an interaction between the helicity vectors and the metric; a convective term depending on the wavevectors and γ-matrices and at last a pure spin term connected only with γ-matrices.

The organization of the work is as follows: in section 2 the phase space representations of the charge current and the canonical energy-momentum tensor of a spin-$\frac{1}{2}$ particle interacting with an external electromagnetic field are obtained. In section 3 the phase space representation of the spin tensor is studied. In section 4 a possible interpretation of the partition of the phase space current into two strictly positive parts is suggested. It is shown that the formalism of relativistic representations of the current and the energy-momentum tensor into phase space based on the Fourier expansion of the wavefunction and an application of the switching procedure [35] in special relativity lead to a new possibility of describing the quantum objects as composed of bradyons, tachyons, luxons and their antiparticles. The fact that the relativistic generalization of the Wigner distribution function in phase space yields a unified description of bradyons and tachyons has already been pointed out by Krüger [38].

2. Phase space representation of the current and the energy–momentum tensor

The interaction between a field $\Psi(x)$ with spin-$\frac{1}{2}$ and an external electromagnetic field $A_\mu(x)$, $\mu=0, 1, 2, 3$, is described by the Dirac equation [41]

$$[\gamma^\mu(\mathrm{i}\partial_\mu+eA_\mu)-m]\Psi(x)=0 \tag{2.1}$$

where γ^μ are the Dirac 4×4 matrices, e and m are the elementary charge and the rest mass of the electron, $\partial_\mu=\partial/\partial x^\mu$. Here and in what follows we use the system of units in which $\hbar=c=1$.

The charge current of the spin-$\frac{1}{2}$ field is given by

$$J^\mu(x)=e\bar{\Psi}(x)\gamma^\mu\Psi(x) \tag{2.2}$$

where $\bar{\Psi}=\Psi^+\gamma^0$, and Ψ^+ is the Hermitian conjugate of the spinor Ψ.

In what follows, it will be useful to apply the Gordon's decomposition [42] of the current:

$$J^\mu(x)=\mathrm{i}e\{\bar{\Psi}(x)\overleftrightarrow{\partial}^\mu\Psi(x)-\partial_\nu[\bar{\Psi}(x)\sigma^{\nu\mu}\Psi(x)]\}/2m. \tag{2.3}$$

The first term in equation (2.3) is the convective, while the second one represents the spin contribution to the current [29].

In the last formula the double arrow indicates a differentiation to the right minus a differentiation to the left and

$$\sigma^{\mu\nu}=\mathrm{i}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)/2 \tag{2.4}$$

is the spin antisymmetric tensor.

Here we shall exploit the symmetric form of the canonical energy-momentum tensor of the spinor field [43]:

$$\begin{aligned}T^{\mu\nu}(x)=-\{&\bar{\Psi}\gamma^\mu(\mathrm{i}\overleftrightarrow{\partial}^\nu-eA^\nu)\Psi+\bar{\Psi}\gamma^\nu(\mathrm{i}\overleftrightarrow{\partial}^\mu-eA^\mu)\Psi-\bar{\Psi}(\mathrm{i}\overleftarrow{\partial}^\mu-eA^\mu)\gamma^\nu\Psi\\&-\bar{\Psi}(\mathrm{i}\overleftarrow{\partial}^\nu-eA^\nu)\gamma^\mu\Psi\}/4\end{aligned} \tag{2.5}$$

where $\vec{\partial}$ and $\overleftarrow{\partial}$ denote the derivatives that act to the right and to the left, respectively.

In order to obtain the phase space representations of (2.2), (2.3) and (2.5) we need the Fourier expansion of the wavefunction

$$\Psi(x)=(2\pi)^{-4}\int \Phi(k)\,\mathrm{e}^{ikx}\,\mathrm{d}^4k \tag{2.6}$$

where $\Phi(k)$ is the Fourier transform of $\Psi(x)$ and d^4k is the volume element in the space of the wavevectors.

Due to the external field in (2.1) it is easy to see that the wavevectors k_μ participating in (2.6) do not lie on the mass hyperboloid, i.e. one has that $(k_\mu+eA_\mu)(k^\mu+eA^\mu)=m^2$. The fact that in such a case it is not necessary to impose any mass-shell restriction on the four-vectors k_μ was especially underlined in [29] and [33].

Applying equation (2.6) to the wavefunctions in the current (2.2) and the energy-momentum tensor (2.5) and omitting the integration we obtain the following phase space densities:

$$J^\mu(x,k)=e(2\pi)^{-4}\bar{\Psi}(x)\gamma^\mu\Phi(k)\,\mathrm{e}^{ikx}=e(2\pi)^{-4}\bar{\Psi}(k)\gamma^\mu\Psi(x)\,\mathrm{e}^{-ikx} \tag{2.7}$$

and

$$\begin{aligned}T^{\mu\nu}(x,k)=(2\pi)^{-4}\{&\bar{\Psi}(x)\gamma^\mu(k^\nu+eA^\nu)\Phi(k)\,\mathrm{e}^{ikx}+\bar{\Psi}(x)\gamma^\nu(k^\mu+eA^\mu)\Phi(k)\,\mathrm{e}^{ikx}\\&+\bar{\Phi}(k)(k^\mu+eA^\mu)\gamma^\nu\Psi(x)\,\mathrm{e}^{-ikx}+\bar{\Phi}(k)(k^\nu+eA^\nu)\gamma^\mu\Psi(x)\,\mathrm{e}^{-ikx}\}/4.\end{aligned} \tag{2.8}$$

Comparison of (2.7) and (2.8) shows that

$$T^{\mu\nu}(x,k)=(2e)^{-1}\{(k^\mu+eA^\mu)J^\nu(x,k)+(k^\nu+eA^\nu)J^\mu(x,k)\}. \tag{2.9}$$

Hence, one can see that the phase space energy-momentum tensor appears as a symmetrized product of the phase space current (2.7) and the canonical momentum

$$\pi^\mu=k^\mu+eA^\mu \tag{2.10}$$

of some kind of object that will be specified later.

Using equation (2.3) we can obtain another form for the phase space representation of the current

$$\begin{aligned}&J^\mu(x,k)=-2eF_T(x,k)k^\mu-e(2\pi)^{-4}\\&\{\bar{\Phi}(k)k_\lambda\sigma^{\lambda\mu}\Psi(x)\,\mathrm{e}^{-ikx}-\bar{\Psi}(x)k_\lambda\sigma^{\lambda\mu}\Phi(k)\,\mathrm{e}^{ikx}\}/2m\end{aligned} \tag{2.11}$$

where

$$F_T(x,k)=\mathrm{Re}[\bar{\Psi}(x)\Phi(k)\,\mathrm{e}^{ikx}] \tag{2.12}$$

is the scalar Terletsky distribution function corresponding to $\bar{\Psi}(x)\Psi(x)$.

From (2.11) it is obvious that the convective part of the phase space current depends on the wavevectors k_μ while the spin part depends on the helicity four-vectors $k_\lambda\sigma^{\lambda\mu}$.

The above observation shows that in the case of a field with spin-$\frac{1}{2}$ the scalar distribution function (2.12) is not sufficient to describe the behaviour of the quantum object in phase space. The currents (2.7) or (2.11) prove to be more appropriate for the purpose.

Now, inserting (2.11) in (2.9) we obtain the following partition of the phase space energy-momentum tensor:

$$T^{\mu\nu}(x,k)=T^{\mu\nu}_{kk}(x,k)+T^{\mu\nu}_{Ak}(x,k)+T^{\mu\nu}_{k\sigma}(x,k)+T^{\mu\nu}_{A\sigma}(x,k). \tag{2.13}$$

The first term in this equation possesses the structure of a pure matter tensor [44], however, with density being the scalar Terletsky distribution function (2.12), i.e.

$$T^{\mu\nu}_{kk}(x,k)=-F_T(x,k)k^\mu k^\nu/m. \tag{2.14}$$

Hence, we have managed to separate from the total phase space energy-momentum density the part that manifests the corpuscularity of the considered object.

The second term in (2.13) shows that the external electromagnetic field acts directly on the wavevectors:

$$T^{\mu\nu}_{Ak}(x,k)=-eF_T(x,k)\,(A^\mu k^\nu+A^\nu k^\mu)/2m. \tag{2.15}$$

The third term in (2.13) describes the contribution from the interaction between the helicity and the wavevectors:

$$\begin{aligned}T^{\mu\nu}_{k\sigma}(x,k)=(2\pi)^{-4}\{&\bar{\Psi}(k^\mu k_\lambda\sigma^{\lambda\nu}+k^\nu k_\lambda\sigma^{\lambda\mu})\Phi\,\mathrm{e}^{ikx}\\&-\bar{\Phi}(k^\mu k_\lambda\sigma^{\lambda\nu}+k^\nu k_\lambda\sigma^{\lambda\mu})\Psi\,\mathrm{e}^{-ikx}\}/4m.\end{aligned} \tag{2.16}$$

The last term is connected with the interaction between the external electromagnetic field and the helicity vectors:

$$\begin{aligned}T^{\mu\nu}_{A\sigma}(x,k)=e(2\pi)^{-4}\{&\bar{\Psi}(A^\mu k_\lambda\sigma^{\lambda\nu}+A^\nu k_\lambda\sigma^{\lambda\nu})\Phi\,\mathrm{e}^{ikx}\\&-\bar{\Phi}(A^\mu k_\lambda\sigma^{\lambda\mu}+A^\nu k_\lambda\sigma^{\lambda\mu})\Psi\,\mathrm{e}^{-ikx}\}/4m.\end{aligned} \tag{2.17}$$

3. Spin tensor in phase space representation

According to [30] the spin tensor that is conserved when the interaction is absent has the following form

$$S^{\mu\nu\lambda}(x)=\bar{\Psi}(x)\{\sigma^{\mu\nu}-(\gamma^\mu\overleftrightarrow{\partial}^\nu-\gamma^\nu\overleftrightarrow{\partial}^\mu)/2m\}\gamma^\lambda\Psi(x)/4+\text{h.c.} \tag{3.1}$$

Applying the well known relation between γ-matrices and the metric tensor

$$\gamma^\nu\gamma^\lambda+\gamma^\lambda\gamma^\nu=2g^{\nu\lambda} \tag{3.2}$$

the spin tensor (3.1) can be transformed into

$$\begin{aligned}S^{\mu\nu\lambda}(x)=\mathrm{i}\{&J^\mu(x)g^{\nu\lambda}-J^\nu(x)g^{\mu\lambda}-\bar{\Psi}(\gamma^\mu\gamma^\lambda\gamma^\nu-\gamma^\nu\gamma^\lambda\gamma^\mu)\Psi/2\\&-\bar{\Psi}(\gamma^\mu\partial^\nu-\gamma^\nu\partial^\mu(\gamma^\lambda\Psi/2m\}/(4e)+\text{h.c.}\end{aligned} \tag{3.3}$$

Setting the expansion (2.6) for $\Psi(x)$ in (3.3), omitting the integration and using the representation (2.11) for the current, one obtains that the phase spin tensor divides also into four terms

$$S^{\mu\nu\lambda}(x,k)=S^{\mu\nu\lambda}_{kg}(x,k)+S^{\mu\nu\lambda}_{\sigma g}(x,k)+S^{\mu\nu\lambda}_{k\gamma}(x;k)+S^{\mu\nu\lambda}_{\gamma}(x,k). \tag{3.4}$$

Here, the first term describes the influence of the metric and the wavevectors on the spin of the particle and depends on the scalar Terletsky distribution function:

$$S^{\mu\nu\lambda}_{kg}(x,k)=-F_T(x,k)(k^\mu g^{\nu\lambda}-k^\nu g^{\mu\lambda})/2m. \tag{3.5}$$

The second term demonstrates the contribution to the spin from the interaction between the helicity vectors and the metric:

$$S^{\mu\nu\lambda}_{\sigma g}(x,k) = -(2\pi)^{-4}\{\bar{\Psi}(k_\tau\sigma^{\tau\mu}g^{\nu\lambda} - k_\tau\sigma^{\tau\mu}g^{\mu\lambda})\Phi\, e^{ikx} + \bar{\Phi}(k_\tau\sigma^{\tau\mu}g^{\nu\lambda} - k_\tau\sigma^{\tau\nu}g^{\mu\lambda})\Psi\, e^{-ikx}\}/8m + \text{h.c.} \tag{3.6}$$

The third term is the convective part of the spin since it depends on the wavevectors in the following way:

$$S^{\mu\nu\lambda}_{k\gamma}(x,k) = -i(2\pi)^{-4}\{\bar{\Psi}(\gamma^\mu k^\nu\gamma^\lambda - \gamma^\nu k^\mu\gamma^\lambda)\Phi\, e^{ikx} + \bar{\Phi}(\gamma^\mu k^\nu\gamma^\lambda - \gamma^\nu k^\mu\gamma^\lambda)\Psi\, e^{-ikc}\}/8m + \text{h.c.} \tag{3.7}$$

The fourth term shows the pure contribution to the spin from the γ-matrices:

$$S^{\mu\nu\lambda}_{\gamma}(x,k) = -i(2\pi)^{-1}\{\bar{\Psi}(\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\nu\gamma^\mu\gamma^\lambda)\Phi\, e^{ikx}\}/8 + \text{h.c.} \tag{3.8}$$

4. Discussion

Exploiting the relativistic quantum field invariants one assumes that the 'fundamental particles are regarded as extended objects' [48], i.e. as objects with internal structure.

We next attempt to specify the possible internal constituents of a spin-$\frac{1}{2}$ particle.

It is necessary here to define the terminology that will be used. If the current, the energy-momentum tensor and the spin tensor are defined in position or in momentum space separately, they describe certain quantum objects. When the same physical quantities are determined in phase space by a distribution function they will describe some kind of hypothetical sub-quantum objects. Some of the peculiar properties of these sub-quantum objects will be discussed below.

In section 2 we have shown that the phase space energy-momentum tensor can be cast into a symmetrized product of the canonical momentum and the phase space current (2.9). This new and specific structure of the energy-momentum tensor turns out to be very useful for interpretation.

Because of the fact that the current $J(x,k)$ is not positive everywhere, using the proposal (1.2), one can introduce two strictly positive current densities in the following way:

$$J^\nu_\pm(x,k) = \{|J^\nu(x,k)| \pm J^\nu(x,k)/2 \geqslant 0. \tag{4.1}$$

It is clear that

$$J^\nu(x,k) = J^\nu_+(x,k) - J^\nu_-(x,k). \tag{4.2}$$

Hence, the energy-momentum tensor in phase space can also be separated into two parts:

$$T^{\mu\nu}_\pm(x,k) = (2e)^{-1}\{(k^\mu + eA^\mu)J^\nu_\pm(x,k) + (k^\nu + eA^\nu)J^\mu_\pm(x,k)\}. \tag{4.3}$$

Modifying the considerations of Pavšič and Recami [45] we accept their terminology and associate 'constituents' with $J^\nu_+(x,k)$ and $T^{\mu\nu}_+(x,k)$ and 'anti-constituents' with $J^\nu_-(x,k)$ and $T^{\mu\nu}_-(x,k)$. As has been pointed out in [46] the availability of a current in phase space corresponds to the presence of a flux density in that space. Hence, the representations (4.1)–(4.3) make it possible to assume the existence of at least two fluxes of 'constituents' and 'anti-constituents' which form the internal ingredients of a spin-$\frac{1}{2}$ particle in the hypothetical sub-quantum level. By sub-quantum level we mean here the level that precedes the quantum one, i.e. it is connected with the internal structure of any elementary quantum object.

One of the peculiarities of the above introduced sub-quantum entities consists in the fact that the well known physical quantities are constructed as differences of the 'constituents' and 'anti-constituents', e.g.

$$T^{\mu\nu}(x,k)=T^{\mu\nu}_{+}(x,k)-T^{\mu\nu}_{-}(x,k). \tag{4.4}$$

Thus one can say that each 'anti-constituent' (or a group of 'anti-constituents' *annihilates* or *nullifies* (figuratively said 'eats up') the action of a corresponding 'constituent' [47]. The remaining uncompensated 'constituents' determine the observed energy-momentum of the quantum object

$$P^{\mu}=\int (T^{\mu o}_{+}(x,k)-T^{\mu o}_{-}(x,k))\,\mathrm{d}^4k\,\mathrm{d}^3x. \tag{4.5}$$

Next, a further differentiation of the sub-quantum entities will illustrate their strange properties. As no mass-shell restriction is imposed on the wavevectors taking part in the Fourier expansion (2.6) it is possible to find time-like objects ($k^{\sigma}k_{\sigma}>0$)—bradyons; light-like objects ($k^{\sigma}k_{\sigma}<0$)—luxons and space-like objects ($k^{\sigma}k_{\sigma}<0$)—tachyons in the 'constituents'/'anti-constituents'. For details concerning this classification see [48].

Since the Fourier decomposition (2.6) contains wavevectors with positive energy ($k_{o}=E/hc>0$) as well as such with negative energy ($k_{o}<0$) as we shall apply the *reinterpretation principle* or *switching procedure* in special relativity [48–50].

It is a well known fact that the spinor current (2.2) is CPT-invariant [51]. Hence the operation of reflecting spacetime (PT) is equivalent to a charge conjugation (C) of this current. The latter inverts the sign of the charge ($j_{o}\to -J_{o}$) and the former reverses the signs of the space and time coordinates $x^{\mu}\to -x^{\mu}$, $\mu=0,1,2,3$. However, the space time reversal applied to the Fourier transform (2.6) is equivalent to a charge in the signs of the wavevectors: $x^{\mu}\to -x^{\mu}\Leftrightarrow k_{\mu}\to -k_{\mu}$ [50]. Hence, it is clear that 'Negative energy objects travelling forward in time do not exist; any negative energy object P travelling backwards in time can and must be described as its anti-object $\bar{P}$ going the opposite way in space (but endowed with positive energy and motion forward in time)' [50].

From the above considerations one concludes that the 'constituents'/'anti-constituents' contain also anti-bradyons, anti-luxons and anti-tachyons.

5. Conclusions

The possibility of transforming the quantum field invariants into quantities in phase space is interpreted as an opportunity for introducing certain kinds of hypothetical sub-quantum entities from which the quantum objects are constituted. Otherwise, we suppose that the above introduced phase space description is related to the internal structure of the quantum particles.

As has been pointed out in [52] 'in quantum mechanics observables are positive-operator-valued measures'. In order to comply with this positiveness requirement we have separated the phase space densities introduced in section 2 into two strictly positive non-equal representatives (see e.g. (1.2), (4.2) and (4.4)). They have been associated with entities called 'constituents and 'anti-constituents'. The novelty is that all typically quantum observables (like (4.5)) have to be constructed as certain differences of the strictly positive phase space quantities. For example, in the case considered, we have represented a spin-$\frac{1}{2}$ particle as composed of the difference of two non-equivalent fluxes that contain bradyons, luxons, tachyons and their anti-particles.

The conclusion that in the case of a spinor field, unlike the case when one deals with a scalar field [30, 36, 53, 54], the scalar distribution function (2.12) is not sufficient to describe the effects of the field, is in agreement with the latest investigations of the matter [32]. It is necessary to consider the components of the four-vector of the charge current (2.7) or (2.11) as distribution functions in the spinor case, or, as was pointed out in [28–30], a 4×4 matrix as a distribution function for mixed states in the phase space.

Finally, we would like to underline that the proposed description is a hypothetical one and is connected mainly with the continuous character of the quantum objects. Their corpuscularity is manifested only by the presence of the pure matter tensor (2.14) in the total phase space energy-momentum tensor (2.13).

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